The multidisciplinary complexity in human population dynamics modeling

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Why so many questions

about human populations ?

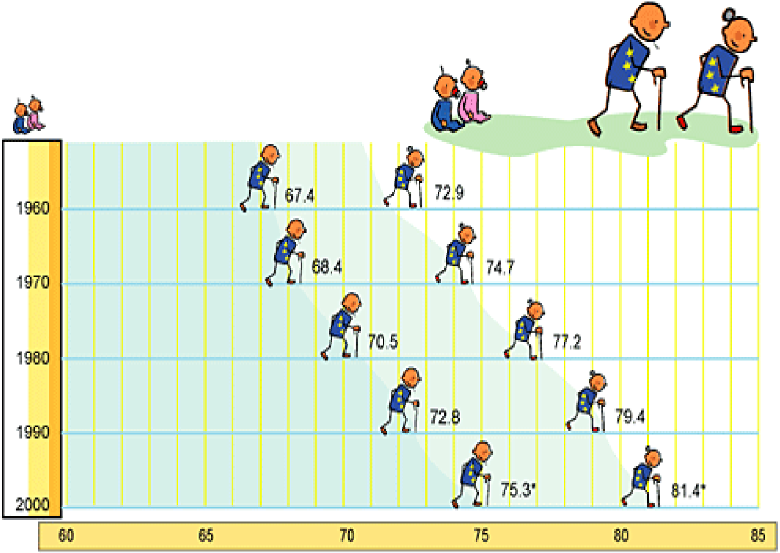


Figure: Longevity Risk

A historic "demographic transition"

Growth of world population

- The past two centuries ($\sim$ 6 generations) presented major changes in the world population, mostly due to a reduction in mortality and fertility
  - From 1 billion in 1800, to more than 7 billion today.
  - Life expectancy at birth has increased by 40 years over the last 150 years.
  - Fertility declines specially in developed countries

Multiple societal, economic, scientific, international challenges

- Population ageing and uncertainty on longevity risk imply
  - political, public health, pension evolutions (govies, private sector)
  - Impacts on the democracy of generational imbalance, climat...
- Important tradition of data collection
  - Villermé(1830), GRO ($\sim$ 1850). (Highlight of past)
  - Now, multiple databases available (INED, UN, WHO, HMD, ...).
  - More than 50 reports /year by public and private institutions.

But, recent breaks in the mortality evolution, showing a need for new models

Aging populations: new phenomenon, without past historical reference

- viability of shared collective systems, in particular (state or private) pension systems
- new generational equilibrium
- role and place of aging population in the society

Complex phenomenon, multi-causes

- difficult to model.
- role of age
- heterogenity

Complex estimation

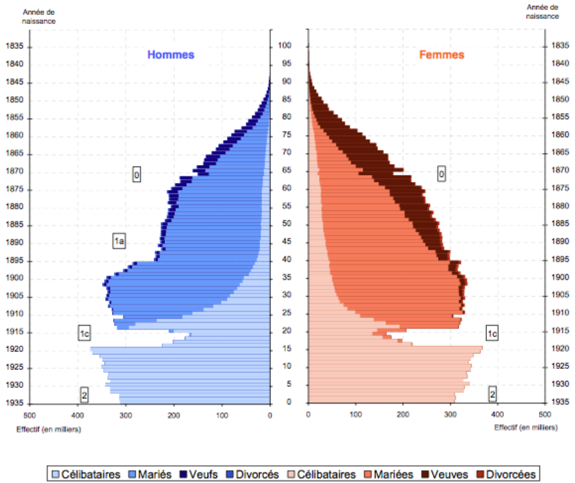
- Coherence of the data
- Age, cohort, period

Age pyramid in France in 1936

National Population by sex, age and matrimonial status

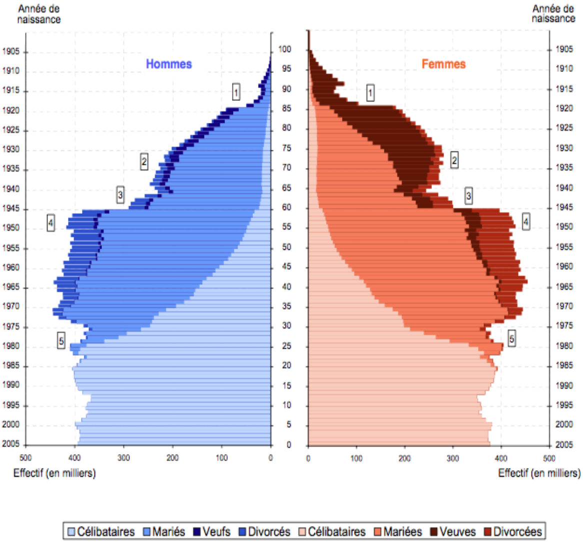
in ascending order of intensity of color: *single, maried, widovers, divorced*,

GRAPHIQUE 1 - REPARTITION DE LA POPULATION TOTALE PAR SEXE, AGE ET ETAT MATRIMONIAL AU 1ER JANVIER 1936



Age pyramid in France in 2006

GRAPHIQUE 1 - RÉPARTITION DE LA POPULATION TOTALE PAR SEXE, ÂGE ET ÉTAT MATRIMONIAL AU 1ER JANVIER 2006



Age Pyramid in France in 1936 / in 2006

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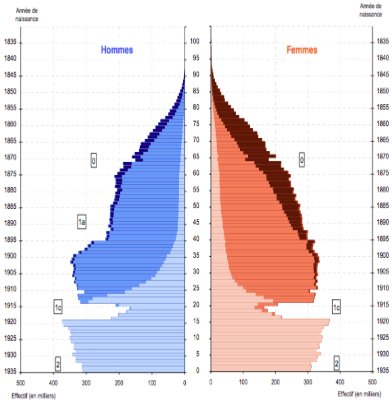

□Célibataires ■Mariés ■Veufs ■Divorcés □Célibataires ■Mariées ■Veuves ■Divorcées

0 Déficit des naissances dû à la guerre de 1870-1871 (classes creuses)

1a Pertes de la guerre de 1914-1918

1c Déficit des naissances dû à la guerre de 1914-1918 (classes creuses)

2 Passage des classes creuses à l'âge de la fécondité

Champ : Population présente en France métropolitaine (territoire actuel)

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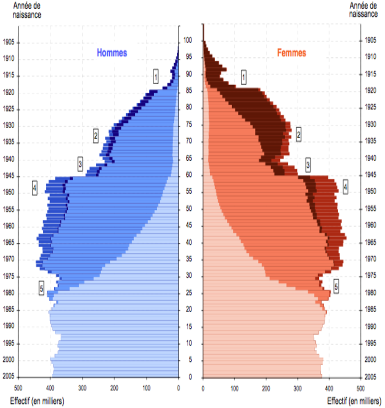


□Célibataires ■Mariés ■Veufs ■Divorcés □Célibataires ■Mariées ■Veuves ■Divorcées

1 Déficit des naissances dû à la guerre de 1914-1918 (classes creuses)

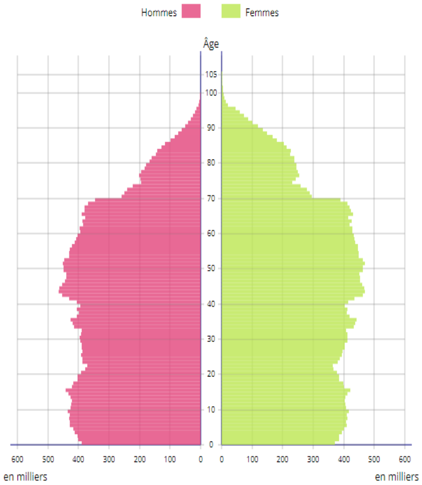
2 Passage des classes creuses à l'âge de la fécondité

3 Déficit des naissances dû à la guerre 1939-1945

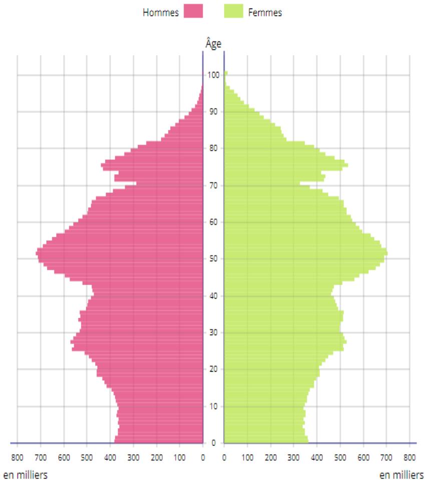
4 "Baby-boom"

5 Fin du "baby-boom"

Champ : France métropolitaine (territoire actuel)



Source : Insee, estimations de population (estimations provisoires arrêtées à mi-avril 2017).



Source : Eurostat, extraction des données le 11 avril 2017.

Demography

"Science" of the populations ?

An Historical Insight

by Hervé Le Bras, (2013)

The three demographies

Three crucial dates in demography

- 1661: Graunt Book, *Natural and Political Observations*
  - the first statistical death table by years and causes in London
  - the first life tables
- 1825: B.Gompertz (Insurance Cie) Mathematical law for the age-related mortality rate, still valuable (also true in biology)
- 1907: A.Lotka (Insurance Cie) Founding the theory of stable human population by introducing the first:
  - the fecondity rate by age of women was added jointed to life tables to give the growth rate of the population;
  - the concept of stable population, where if the growth rate is nul, the population is the product of yearly births with life expectancy;
  - still the actual basis for indirect methods of demographic projection.
- 1927: International Labour Office Meeting Creation of the International Union for the Scientific Study of Population (IUSSP)

The three components of demography

Statistical demography and mathematical demography

- In charge of the constitution of Data Bases
- Study of demographic parameters, mortality rate, life tables,
- Sensitive to political views
- Theoretical mathematics offer models for projections

Political Demography

- Central concept of fertility (Malthus, negative eugenics,..)
- Creation of international structures (IUSSP)
- Institute, Inst Nat SEE, Inst Nat Etudes Demographiques.

Mixing of the three components

- Developed countries are concerned with population (M.Foucault, 2004)
- Methods for projections in the future by age component
- Recently, Big Data without model

Cholera Outbreaks in the 19th Century

Cholera outbreaks in France and England

- 1831, 1848 – 1854, 1866 – 1867 and 1888 – 1889
- 102.000 deaths in France in 1832 and 143.000 in the 1850*s* (Total population 36 millions)
- 6536 deaths in 1831 in London and 14137 deaths in 1848 – 1849

Cholera in England

- Need of public measures
- Creation in 1836 of the General Register Office (G.R.O) to centralize vital statistics
- Data collected in 2193 districts, by competent registers. Help to control the epidemics
- The Doctor Snow identified the importance of water on the diffusion of the Cholera

The Cholera in France and on an International scale

Cholera in France

- Same Need of public measures : Paris 2% population
- Iconic example of first links between Medicine and Statistics
- Data are collected at the department level (Napoleon Structure)
- Less efficient but compensated by the High level of the french medicine

Cholera Pandemic and International Health Organization

- Industrial Revolution, in particular in transport, Canal de Suez,
- Ten International Conferences to define rules on international travels of control
- Final consequence: creation of the World Health Organization (1945).

Aggregate mortality indicators

Life expectancy at birth

- Lifetime of an individual (random): τ
- Life expectancy at birth: $\mathrm{E}[\tau]$, at ten $\mathrm{E}[\tau - 10/\tau > 10]$

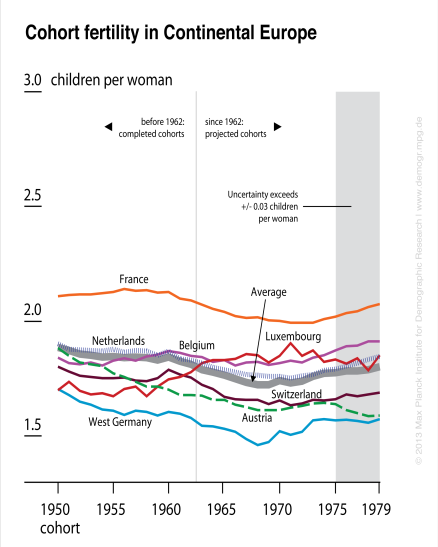
Death rate

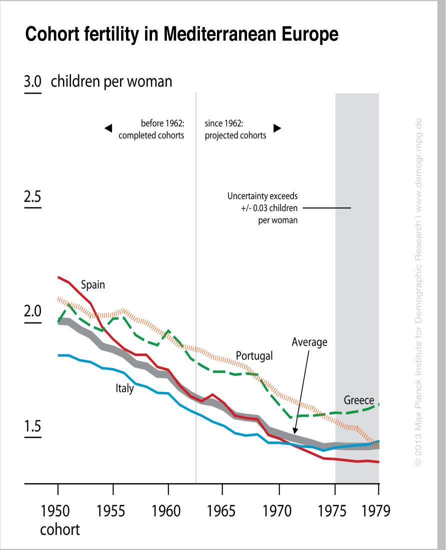
- Death rate $d(a)$, such that $\mathrm{P}(\tau > a) = e^{-\int_0^a d(s)ds}$
- In practice, annual death probability reduction
  $q(a) = \mathrm{P}(\tau < a + 1 \mid \tau \geqslant a)$
- Mortality plateau (old age)

Fertility rate

- Complex notion
- With large political connotation (fertility, immigration)

Fertility rate in Europe/1950-80





Heterogeneity
and Aggregated Human populations

Taking into account heterogeneity

Heterogeneity and estimation or forecast of mortality rates:

- Large amount of data at the national level → lower variance.
- **But** bigger populations imply greater heterogeneity.
- Trade-off:
  - Behaviors deviate further from average → increase of variance.
  - Heterogeneity of the population changes over time.

Consequences

- Not taking into account heterogeneity can lead to:
  - Increased inequalities due to public health reforms (A.Bajekal et al. (2017)) or redistribution properties of state pensions.
  - Bias in computation of regulatory reserves.
- Better understanding of heterogeneity allows for comparison between populations of far different compositions:
  Ex:Basis risk exposure for pension funds, due to the difference between the national and the insured populations.

A shift in paradigm in human populations...

Diverging trends in longevity documented at multiple levels:

- Countries with *similar* mortality level up the 1980s, now diverge
  - Gaps in female life expectancy at age 50 in 10 high-income countries:
  - $\leqslant 1$ year in 1980, $\geqslant 5$ years in 2007,(source: HMD).
- *Widening* of socioeconomic and geographical mortality inequalities:
  - Gap in male life expectancy at age 65 between higher managerial positions and routine occupations (England-Wales):
  - 2.4 years 1982-1986, 3.9 years 2007-2011, (source:ONS)

Comment of National Research Council (2011):

> *"What is perhaps more surprising is that large differences [...] began relatively abruptly around 1980, and that it has taken so long for this divergence to be recognized and analyzed."*

English Databases, Sarah+Heloise

Two databases/New

- 1981-2007: Department of Applied Health Research, UCL.
- 2001-2015: Office for National Statistics, released April 2017).

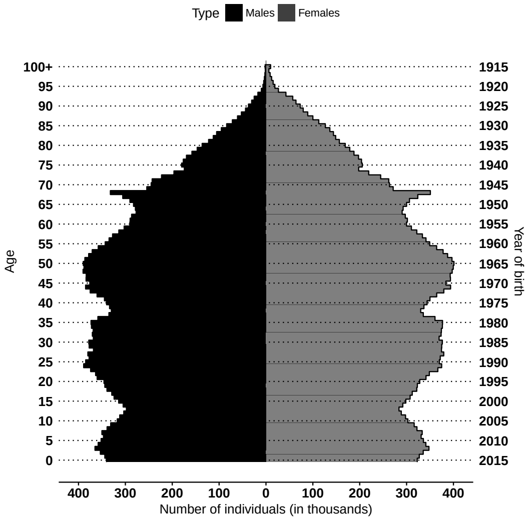
- English cause-specific number of deaths and mid-year population estimates per socioeconomic circumstances, age and gender.

Index of multiple deprivation (IMD)

Socioeconomic circumstances are measured, based on the postcode.

- Small areas (LSOA) are ranked based on seven broad criteria: income, employment, health, education, barriers to housing and services, living environment and crime.
- This ranking makes it possible to divide the population in to 5 quintiles with about the same number of individuals in each quintile.
- Post-code based index serves as SES proxy and captures information on individuals environment.

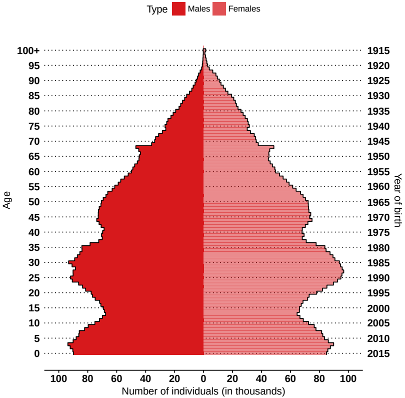
English age pyramid, 2015



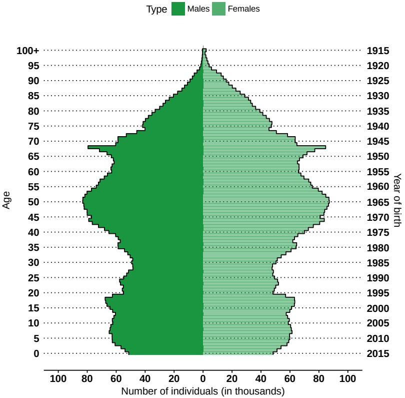
Median age: 39y.

Population composition $(- \rightarrow +)$, 2015

Age-pyramids by IMD quintile, 2015



(a) Most deprived quintile ($)
Median age: 33y

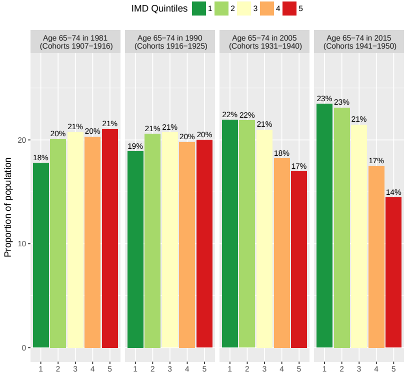


(b) Least deprived quintile ($$$$$)
Median age: 44.2y

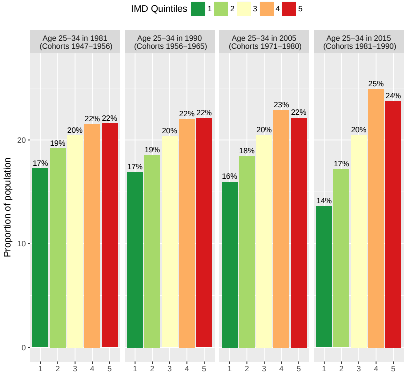
- Baby-boom cohort less deprived than younger/older cohorts.

Changes in the population composition

Figure: Composition of males age classes in years **1981, 1990, 2005, 2015**.



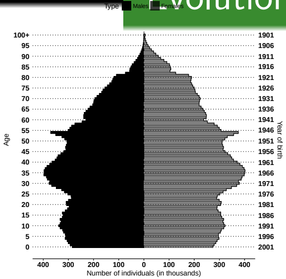
(a) Age class **65-74**



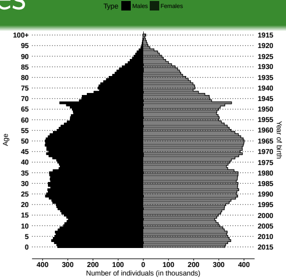
(b) Age class **25-34**

- Decrease of deprivation over time for older age classes, (IMD 1+2: 28% → 46%).
- Increase of deprivation for younger age classes, (IMD 1+2: 36% → 31%).

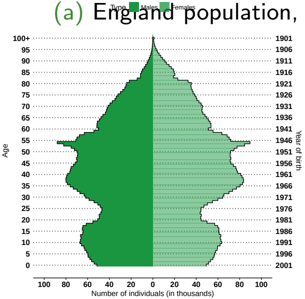
Evolution by classes



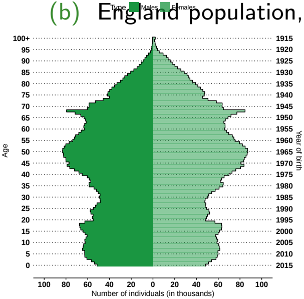
(a) England population, 2001



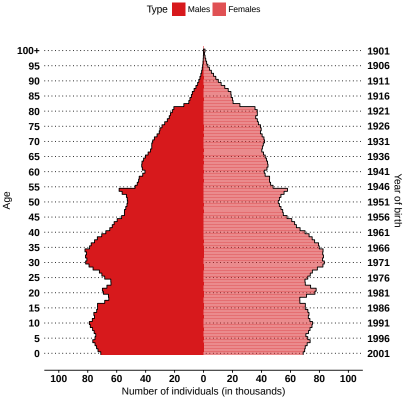
(b) England population, 2015



(c) Least deprived quintile ($$$$$)



(d) Least deprived quintile ($$$$$)



FiguresSylvie/Pyramid_2015_Pauvres

(a) Most deprived quintile ($), 2001 (b) Most deprived quintile ($), 2015

Figure: Age pyramids in 2001 and 2015

Human Population Modeling

The classical mortality rate point of view

Human Mortality Database

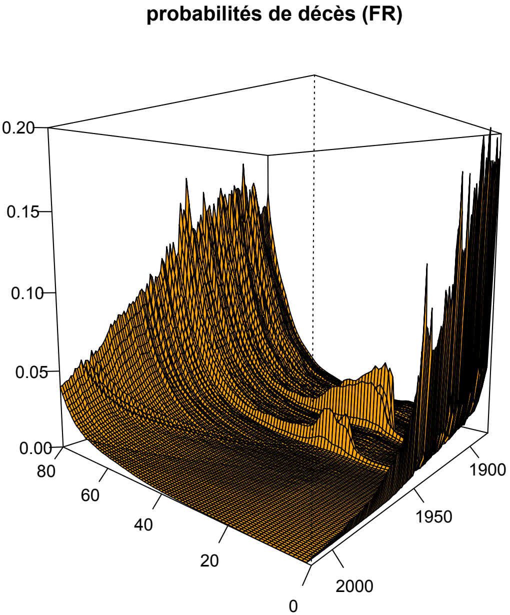
Public Data

- Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany).
- Available at www.mortality.org or www.humanmortality.de

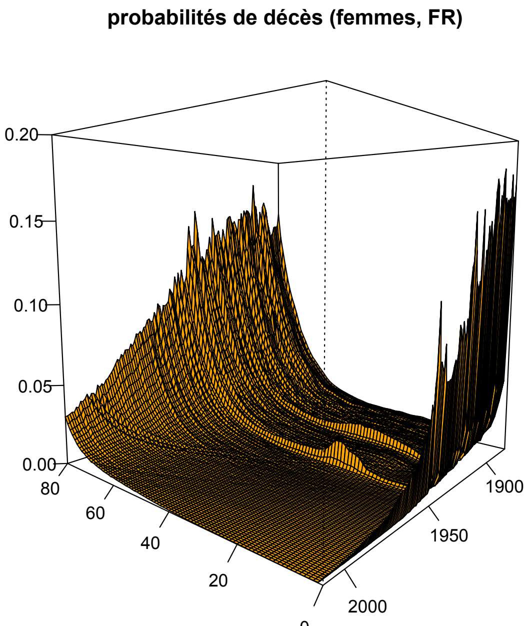
French Data

- INSEE
- Demographic Permanent Sampling

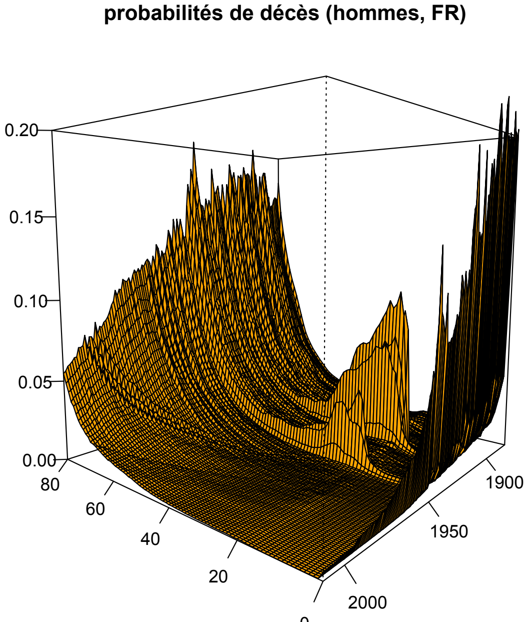
Mortality structure (France)



National mortality by gender (France)



National mortality by gender (France)



National mortality: log q(a,t)

- Looking at $log\ q(a,t)$ age a in [0,100]
- for different years t (1950,1965,1980,1995,2005)

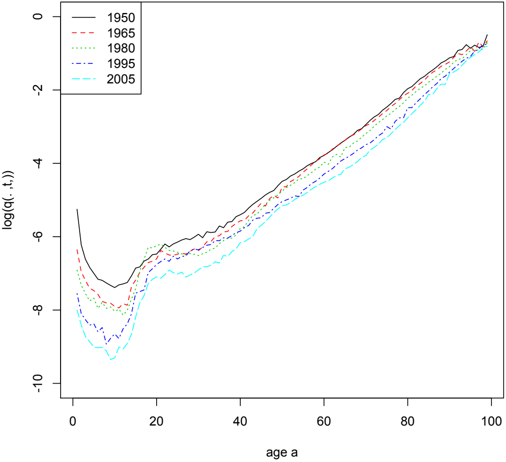


Figure: Logarithm of annual death probabilities (national population)

Logit representation (Cairns-Blake-Dowd)

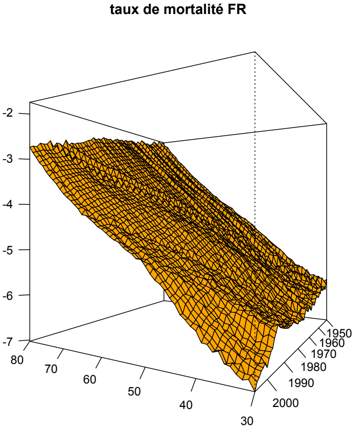


Figure: Logit of annual death probabilities

Cairns-Blake-Dowd (CDB) model

Model for high ages (actuarial perspective) :

$$\mathrm{logit}\big(q(a,t)\big) = \kappa_1(t) + a.\kappa_2(t) + \epsilon_{a,t},$$

- $\kappa_1(t)$: overall reduction in mortality through time, for all ages,
- $\kappa_2(t)$: specific adjustment at each age,
- $\epsilon_{a,t}$ is the residual noise.

$\Rightarrow$ choice of a particular form of age dependency (linear)

$\Rightarrow$ 2 time factors

Estimating parameters: for each year t between 1950 and 2010, we perform the linear regression over ages between 60 and 95, which gives parameters $\kappa_1(t)$ and $\kappa_2(t)$

CBD parameters

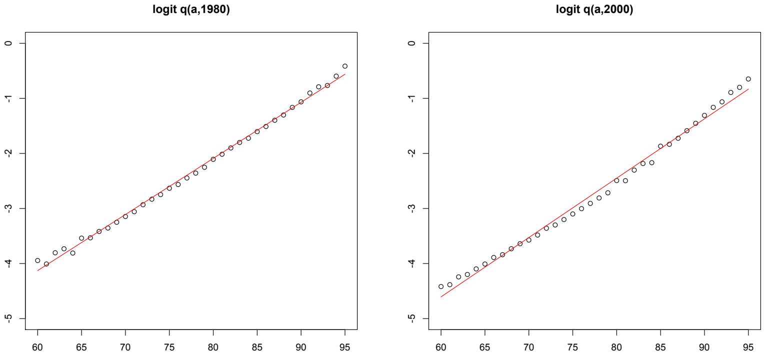


Figure: Logit of annual death probabilities for years 1980 and 2000 (French males)

CDB estimated parameters

Time series κ_1 and κ_2 can be viewed as a fluctuating environment

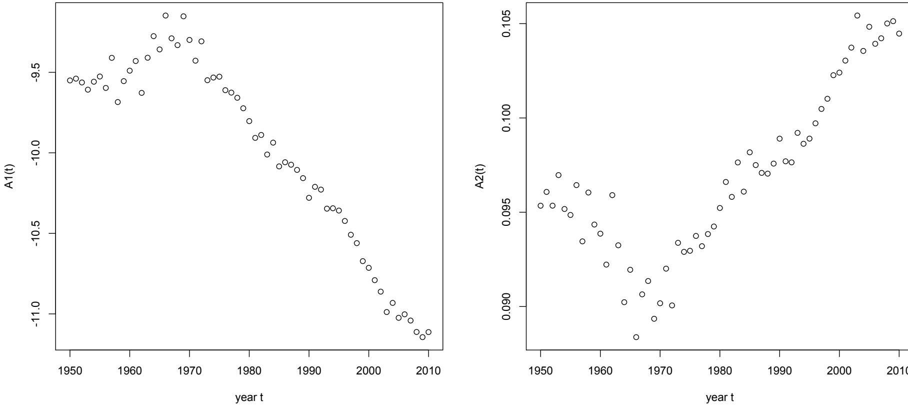


Figure: Processes kappa1 (left) and kappa2 (right) estimated for French males (ages 60-95) between 1950 and 2010

Structured population by traits and age

Modelisation following Sarah presentation

Qualitative properties

- Data show strong dependency from the past
- Generational effect
- Cohort and environmental effect

Mathematical Challenges

- to generalize the known methods to non Markovian case (ex stable convergence)
- to develop efficient algorithms
- "Mescoscopic" scale:
  - Identify the level of aggregation: description of subgroups rather than individual life courses.
  - to be carefull to the neighborhood effect (Hierarchical models?)

Dynamic population structured by age

- The new population becomes $\tilde{\xi}_t = \{(t - T_1, X_1), (t - T_{N_t}, X_{N_t})\}$.
- The mark is now depending on the point of time t by $r_t(a) = (t - a)^+$,
- $\tilde{N}^a_t(A \times B) = N_t(r_t(A) \times B)$ is no more a measure in t,
- although if its expectation $\int_0^t \mathbf{1}_A(t - s) ds m(B) = \int_0^t \mathbf{1}_A(s) ds$

Deterministic formula for counting measure with age

- For differentiable f in age, coupled with integration by parts (formula for the online process)

$$z_t(f) = f(0) z_t(1) + \int_0^t z_v(f') dv.$$

- $z_t(f)$ is of finite variation
- Application to Hawkes process

Thinning equations for Birth processes

The thinning construction can be used to define a wide variety of processes as solution to stochastic equations.

Intensity for linear Birth Process

- Generalisation of Poisson process, but pure jump Markov process on $\mathbb{N}$, (N_t) non decreasing in time with jump 1 and intensity λn
- The time between two jumps is exponential of parameter λn, and independent

Representation as solution of SDE

- Stochastic intensity $\lambda_t N_{t-}$
- Equation $dN_t = \int_{\mathrm{R}_+} \mathbf{1}_{\{\theta \leqslant \lambda_t N_{t-}\}} Q(dt, d\theta), N_0 = x$
- Solution by recursive method starting with the process
  $dX_t^1 = \int_{\mathrm{R}_+} \mathbf{1}_{\{\theta \leqslant \lambda_t x\}} Q(dt, d\theta)$,
  $dX_t^2 = \int_{\mathrm{R}_+} \mathbf{1}_{\{X_{t-}^1 > x\}} \mathbf{1}_{\{\theta \leqslant \lambda_t X_{t-}^1\}} Q(dt, d\theta)$ and so on....

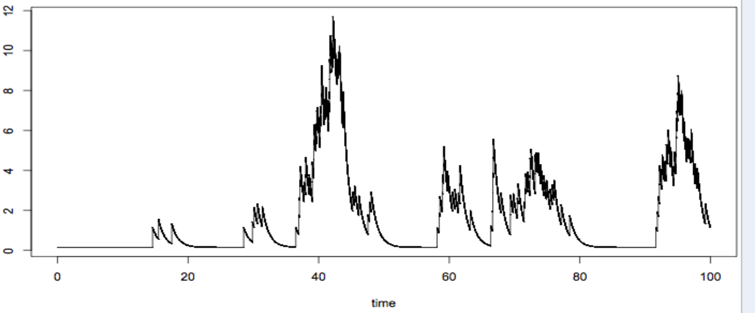
Online Hawkes process

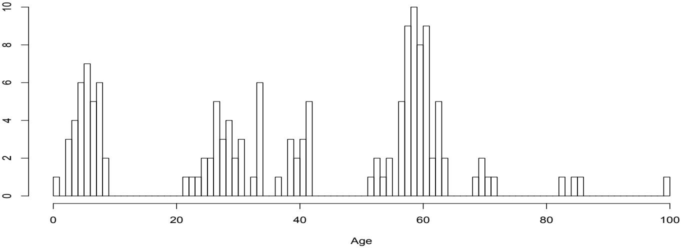
A Linear self-exciting processes

- A point process N with jump times (T_n) and path-dependent intensity λ_t, ($N_t = N_0 + \int_{(0,t]} \lambda_s dt\ + \mathcal{F}$-martingale)
- Hawkes (1971): Linear self-excitation
$$\lambda_t = \bar{\mu} + \int_{(0,t)} \phi(t-s)\mathrm{d}N_s = \bar{\mu} + \sum_{T_n<t} \phi(t-T_n),$$
- ϕ=fertility function

- Simple Hawkes example: Autoregressive point process,
$$\phi(t) = \alpha e^{-\beta t} \qquad \lambda_t = \bar{\mu} + \alpha \int_{(0,t)} e^{-\beta(t-s)}\mathrm{d}N_s$$
- Population fertility function $\phi^2(a) = \mu + k \exp -c(t-t_f)^{+,2}$

Intensity and Age Pyramid for Hawkes





Demographic rates

Demographic rates: an individual of traits $x_t \in \mathcal{X} \subset \mathrm{R}^d$ and age $a_t \in [0, \bar{a}]$ at time t, (born at time 0)

- Dies at rate $d\,(x_t, a_t, t, Y)$
- Gives birth at rate $b\,(x_t, a_t, t, Y)$
  and the new individual has traits $x' \sim K^b(x_t, a_t, dx')$
- Evolves during life at rate $e\,(x_t, a_t, t, Y)$
  from traits x_t to $x' \sim K^e(x_t, a_t, dx')$

Environmental factors

- Demographic rates depend on characteristics, age, time and on the stochastic environment Y
- Conditionally on the environment Y, the events for a given individual are jumps of a counting process

The thinning construction can be used to define a wide variety of processes as solution to stochastic equations.

Intensity for spatial Birth Process

- naissance +"mutation"individual = rate becomes $b(x')k_b(x',x)m(dx')$
- aggregated rate of birth mutation= $$\beta(\xi,x) = \sum_{x'\in\xi} b(x')k_b(x',x)$$
- Equation $Z(dt,dx) = \int_{\mathrm{R}_+} \mathbf{1}_{\theta\leqslant\beta(Z_{t-},x)} Q_b(dt,dx,d\theta)$

Birth with age

- First define the new kernel with the age
- Applied the previous relation to process $d\langle Z_t, f\rangle = \langle I^b_{Z_{t-}} f(.,0), Q_b\rangle(dt) + \langle Z_t, \partial_a f\rangle dt$
- Existence result similar to the linear case

Death process

Fondamental asymmetry

- since the newborn is from outside,
- then the death remove an individual in the population

How to select an individual by its characteristics

- the counting measure on E is not a "Radon" σ-finite measure on E
- Necessity to give a measurable and adapted process to select individual in a given population

Numbering a population and Death process

Envelop process of population path without accumulation

- The age desagregated population $\tilde{\xi}_s(dx) = \xi_s(dx, \mathrm{R}_+)$
- The non decreasing envelop $\bar{\xi}_t = \bigcup_{s\in[0,t]} \tilde{\xi}_s$ process
- $\bar{\xi}_t$ has only finite number of jumps on $[0, t]$, denoted by (S_k)
- (S_k) are also times of jumps for the path ξ_t
- The sequence $(X_k, A_k(.))_{k\geqslant N_0+1}$, $X_k = \bar{\xi}_{S_{k-N_0}} \backslash \bar{\xi}_{S^-_{k-N_0}}$, and $A_k(t) = t - S_k$

Spatial death process

- A Poisson point measure $Q_d(ds, di, d\theta)$ on $\mathrm{R}_+ \times \mathbb{N}^* \times \mathrm{R}_+$
- with intensity measure $q_d(ds, di, d\theta) = ds\, n(di)\, d\theta$
- $I^d(Z_{t-}, i, \theta) = \mathbf{1}_{X_i\in Z_{t-}} \mathbf{1}_{\theta\leqslant d(X_i)}$
- Using the previous numbering, we see that $Z(dt, dx) = -\int_{i\in\mathbb{N}*} \int_{\theta\in\mathrm{R}_+} I^d(Z_{t-}, i, \theta)\delta_{X^i}(dx)Q_d(dt, di, d\theta)$,
- Same transformation with age

Theorem Bezborodov (2014), Garcia 1999, ...

- If $\xi_0^1 \subset \xi_0^2$,
- $\beta_1(x, \eta^1) \leqslant \beta_2(x, \eta^2) \quad \eta^1 \subset \eta^2$
- $d_1(x, \eta^1) \geqslant d_2(x, \eta^2) \quad \eta^2 \subset \eta^1, x \in \eta^1$

The comparison theorem

There exists a cadlag process (η_t) such that $\eta_t \subset \xi_t^2$ having the same law that (ξ_t^1)

Sketch of the proof without age, and swap

$$\eta(dt, B) = \int_{B \times \mathrm{R}^+} \mathbf{1}_{[0, b_1(x, \eta_{s-})]}(\theta) dQ^b(dt, dx, d\theta)$$

$$- \int_{\mathbb{N} \times \mathrm{R}^+} \mathbf{1}_{\{x_i^2 \in \eta_{s-} \cap B\}} \mathbf{1}_{[0, d_1(x_i^2, \eta_{s-})]}(\theta) dQ^d(dt, di, d\theta)$$

Applications of comparison theorem

In progess

Study classical properties of population processes

- Agregation by traits and convexity
- Localisation and explosion
- Monotonic convergence

Stochastic order on the space of configuration

- Starting from the result of Preston (1975) on the stochastic order for the Point random field
- Property of the stochastic order on the distributions of the population processes Z_t in terms of demographic characteristics

The Poisson measures driving the equation

- Q_b, Q_d, Q_e
- $I^b(Z_{t-}, t, x, \theta)$, $I^d(Z_{t-}, t, i, x, \theta)$, $I^e(Z_{t-}, t, i, x, \theta)$

The BSD Population equation

$$\begin{aligned} d\langle Z_t, f\rangle &= \langle I^b_{Z_{t-},t} f(.,0), Q_b\rangle(dt) - \langle I^d_{Z_{t-},t} f(X_., A_.(t)), Q_d\rangle(dt) \\ &+ \langle I^e_{Z_{t-},t}[f(., A_.(t)) - f(X_., A_.(t))], Q_e\rangle(dt) + \langle Z_t, \partial_a f\rangle dt. \end{aligned} \tag{1}$$

Hypotheses, $E = \mathrm{R}^d, m(dx) = l(dx)$

- $\int b(x,\eta)dx \leqslant c_1|\eta| + c_2$
- $\sup_x \sup_{\{|\eta| \leqslant m\}} d(x,\eta) < \infty$

Then, existence and strong uniqueness

Cohort effect

An example of numerical experiment
to explain an observed phenomenon

Cohort effect

Birth cohort

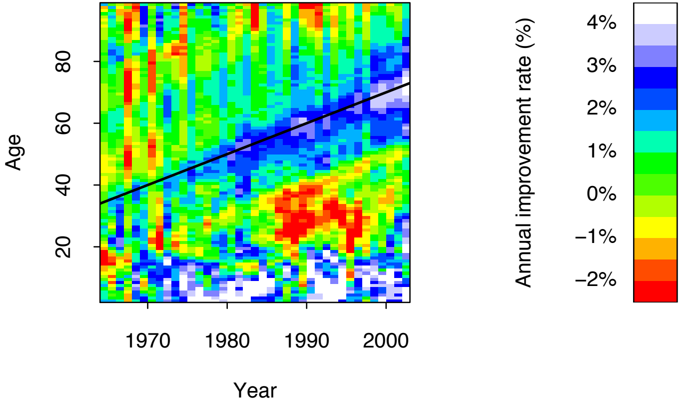
- Birth cohort for the period $[t_1, t_2]$: group of individuals born between t_1 and t_2.
- Individuals of the same birth cohort share similar demographic characteristics ("cohort effect")

Age, Period, Cohort

- Age, Period, Cohort analysis put a lot of problems in practice, in different domains, medecine, sociology,...due to the lag in data,..insurance...
- Huge literature on APC problems

Golden cohort: generations born between 1925 and 1940
Cairns et al. (2009) $r_{a,t} = (q_{a,t-1} - q_{a,t})/q_{a,t}$

The Golden cohort has experienced more rapid improvements than earlier and later generations.



Analysis of R. C. Willets, 2004

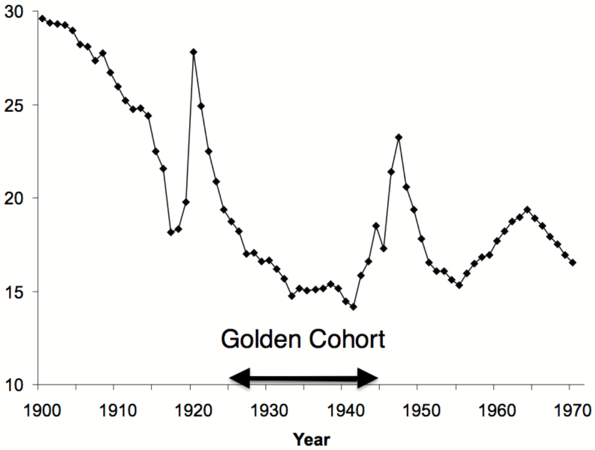
Some possible explanations:

- Impact of World War II on previous generations,
- Changes on smoking prevalence: tobacco consumption in next generations,
- Impact of diet in early life,
- Post World War II welfare state,
- Patterns of birth rates

"One possible consequence of rapidly changing birth rates is that the average child is likely to be different in periods where birth rates are very different. For instance, if trends in fertility vary by socio-economic class, the class mix of a population will change."

The Cohort Effect: Insights And Explanations, 2004, R. C. Willets

Cohort effect and Fertility



Data source: www.mortality.org

Figure 6. Crude birth rate per 1,000 population, England and Wales, 1900 to 1970

Simple toy model

The different rates

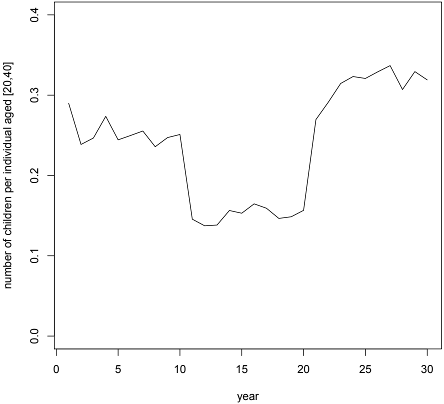
- Reference death rate $\bar{d}(a) = A\exp(Ba)$
- Parameters $A \sim 0.0004$ and $B \sim 0.073$ estimated on French national data for year 1925 to capture a proper order of magnitude
- "Upper class": time independent death rate $d^1(a) = \bar{d}(a)$ and birth rate $b^1(a) = c\mathbf{1}_{[20,40]}(a)$ (c=0.1)
- "Lower class": time independent death rate $d^2(a) = 2\bar{d}(a)$ but birth rate $b^2(a, t) = 4c\mathbf{1}_{[20,40]}(a)\mathbf{1}_{[0,t_1]\cup[t_3,\infty)}(t) + 2c\mathbf{1}_{[20,40]}(a)\mathbf{1}_{[t_2,t_3]}(t)$

Comment Constant death rates but reduction in overall fertility between times t_1 (=10) and t_2 (=20).

- Aim: Test the cohort effect by computing standard demographic indicators on the population

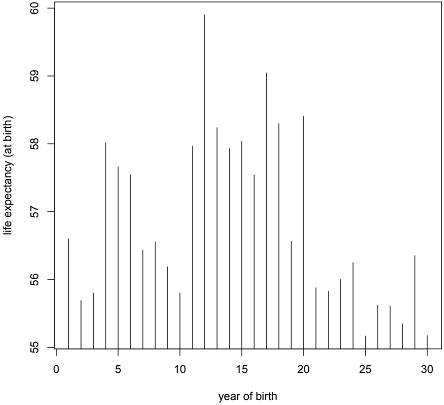
Aggregate fertility

- One trajectory with 20000 individuals (randomly) splitted between groups. Estimation of aggregate fertility



Life expectancy by year of birth

- "Cohort effect" for aggregate life expectancy



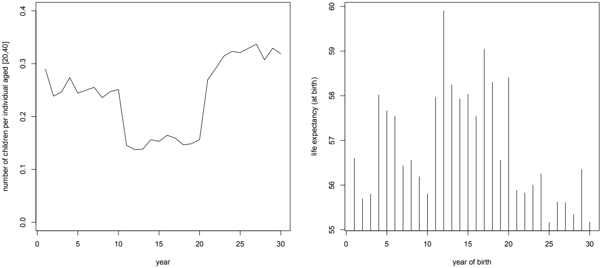


Figure: Observed fertility (left) and estimated life expectancy by year of birth (right)

- Death rates by specific group remain the same
- But reduction in fertility for "lower class" during 10-20 modifies the generations composition
  $\Rightarrow$ "upper class" is more represented among those born between 10 and 20

Analysis the baby-boom as a cohort effect

Cohort effect

- Major transformation of (birth rate) in the population
- During one generation (25 years)
- Associated with increasing wealth, education level..

Major impact on projection in the future

- First step, reproduce this phenomenon
- Analyze the consequences of the stabilization of schooling, and other social progress
- Progress in medecine, and connected health.
- Increasing role of the globalization

Perspectives

The dynamic population model as an opportunity of the experimentation (as a Laboratory

- Analyze of the baby boom period as a cohort effect of a different type by comparing different countries
- Use comparison result to control the impact of data aggregation
- The model allows us to take into consideration interactions between individuals, as in the two sex-pyramids
- Intergenerational aspect

The software of Daphne, Vincent, Sarah

- Use recent development in computer simulation to make the experimentation for efficient
- Thanks to for their remarkable simulation tool
- ref: https://github.com/DaphneGiorgi/IBMPopSim